Shell-model symmetries (solutions)

1. *Pairing interaction in a single-j shell.*

   - The coefficient in front of $[[a_j^\dagger \times a_j^\dagger]^{(0)} \times [\tilde{a}_j \times \tilde{a}_j]^{(0)}]_0^{(0)}$ should be equal to $-\frac{1}{2}v_0 = +\frac{1}{4}(2j+1)g = 0.875$. All others are 0.
   - The spectrum has a $J = 0$ state at -1.75 and all other states at 0.
   - For $n = 4$ there are states at -3 ($v = 0$), -1.25 ($v = 2$) and 0 ($v = 4$). For $n = 6$ there are states at -3.75 ($v = 0$), -2 ($v = 2$), -0.75 ($v = 4$) and 0 ($v = 6$). The seniority of the ground state is $v = 0$ and its angular momentum is $J = 0$. Allowed angular momenta of the seniority $v = 2$ states are $J = 2$, 4, 6, 8, 10 and 12; besides $J = 0$, these are the angular momenta allowed for two identical nucleons in a $j = 13/2$ shell. States at zero energy have seniority $v = n$; their energy is zero because all nucleons are unpaired and therefore non-interacting.
   - For $n = 3$ there are states at -1.5 ($v = 1$) and 0 ($v = 3$). For $n = 5$ there are states at -2.5 ($v = 1$), -1 ($v = 3$) and 0 ($v = 5$). The seniority of the ground state is $v = 1$ and its angular momentum is $J = 13/2$.
   - The eigenvectors coincide with the basis used by `ArbModel` which is therefore the seniority basis.

2. *Delta interaction in a single-j shell.*

   - The two-body matrix elements are $\upsilon_0 = -7G$, $\upsilon_2 = -\frac{112}{65}G$, $\upsilon_4 = -\frac{2268}{2431}G$, $\upsilon_6 = -\frac{28000}{46189}G$, $\upsilon_8 = -\frac{1750}{4199}G$, $\upsilon_{10} = -\frac{27216}{96577}G$ and $\upsilon_{12} = -\frac{30492}{185725}G$. This interaction conserves seniority because

   $$1615\upsilon_2 - 4275\upsilon_4 - 1456\upsilon_6 + 3196\upsilon_8 + 5145\upsilon_{10} - 4225\upsilon_{12} = 0.$$

   - The input coefficients for `ArbModel` are 0.875, 0.481615, 0.349856, 0.273213, 0.214796, 0.161425 and 0.102611 for $J = 0$, 2, 4, 6, 8, 10 and 12, respectively.
   - The two-particle spectrum is -1.75, -0.430769, -0.233237, -0.151551, -0.104191, -0.0704516 and -0.0410446 for $J = 0$, 2, 4, 6, 8, 10 and 12, respectively.
   - The wave function of the $J = 0$ state remains unchanged as compared to that found with the pairing interaction. The ground state has seniority $\upsilon = 0$ and, although the delta interaction is in general *not* analytically solvable, the structure of the ground state is $(S_+)^{n/2}|\mathrm{o}\rangle$ *exactly*.
   - The wave functions of the ground and the first-excited $J = 13/2$ states remain unchanged as compared to those found with the pairing interaction. The ground state has seniority $\upsilon = 1$ and the first-excited state has seniority $\upsilon = 3$.

3. *Seniority isomers in the* $1g_{9/2}$ *shell.*

   - The two-body matrix elements from ^{70}Ni are (in MeV, relative to v_0):

     $$v_0 = 0.000, v_2 = 1.260, v_4 = 2.229, v_6 = 2.678, v_8 = 2.860.$$

     The two-body matrix elements from ^{92}Mo are (in MeV, relative to v_0):

     $$v_0 = 0.000, v_2 = 1.510, v_4 = 2.283, v_6 = 2.612, v_8 = 2.761.$$

   - No because $65v_2 - 315v_4 + 403v_6 - 153v_8 \neq 0$ in both cases.
   - The `ArbModel` input coefficients for ^{70}Ni and ^{72}Ni: 0.000, -1.40872, -3.3435, -4.82783 and -5.89604 for $J = 0$, 2, 4, 6 and 8, respectively. For ^{92}Mo and ^{94}Ru they are: 0.000, -1.68823, -3.42450, -4.70885 and -5.69195.
   - Results for the 6_i^+ and 8_1^+ levels are summarized in the table. The 6_1^+ and 8_1^+ states have seniority $v = 2$; the 6_2^+ and 6_3^+ states have seniority $v = 4$.

Table 1: Energies (in MeV) and B(E2) values (arbitrary units)

	^{72}Ni		^{94}Ru	
	E_x	$B(\text{E2}; 8_1^+ \to 6_i^+)$	E_x	$B(\text{E2}; 8_1^+ \to 6_i^+)$
8_1^+	2.876	—	2.769	—
6_1^+	2.627	0.035	2.592	0.028
6_2^+	2.655	0.740	2.918	0.728
6_3^+	4.050	0.107	4.013	0.111

4. *SU(4) symmetry.*

- For two identical nucleons in $1p_{1/2}$ and $1p_{3/2}$ we have:

$$(1p_{1/2})^2 : J = 0, \quad 1p_{1/2}1p_{3/2} : J = 1, 2, \quad (1p_{3/2})^2 : J = 0, 2.$$

- For a neutron and a proton in $1p_{1/2}$ and $1p_{3/2}$ we have:

$$(1p_{1/2})^2 : J = 0, 1, \quad 1p_{1/2}1p_{3/2} : J = 1^2, 2^2, \quad (1p_{3/2})^2 : J = 0, 1, 2, 3.$$

- For $T = 1$: $(1p_{1/2})^2 : J = 0, \quad 1p_{1/2}1p_{3/2} : J = 1, 2, \quad (1p_{3/2})^2 : J = 0, 2.$
  For $T = 0$: $(1p_{1/2})^2 : J = 1, \quad 1p_{1/2}1p_{3/2} : J = 1, 2, \quad (1p_{3/2})^2 : J = 1, 3.$
- Energy matrices and their eigenvalues corresponding to the different angular momenta J and isospins T:

$$(J = 0, T = 1) \quad [(1p_{1/2})^2, (1p_{3/2})^2] : \begin{bmatrix} -1 & -\sqrt{2} \\ -\sqrt{2} & -2 \end{bmatrix} \Rightarrow -3, 0$$

$$(J = 1, T = 1) \quad [1p_{1/2}1p_{3/2}] : \begin{bmatrix} 0 \end{bmatrix} \Rightarrow 0$$

$$(J = 2, T = 1) \quad [1p_{1/2}1p_{3/2}, (1p_{3/2})^2] : \begin{bmatrix} -\frac{4}{5} & \frac{2}{5}\sqrt{2} \\ \frac{2}{5}\sqrt{2} & -\frac{2}{5} \end{bmatrix} \Rightarrow -\frac{6}{5}, 0$$

$$(J = 1, T = 0) \quad [(1p_{1/2})^2, 1p_{1/2}1p_{3/2}, (1p_{3/2})^2] : \begin{bmatrix} -1 & 0 & \sqrt{\frac{2}{5}} \\ 0 & -2 & -2\sqrt{\frac{2}{5}} \\ \sqrt{\frac{2}{5}} & -2\sqrt{\frac{2}{5}} & -\frac{6}{5} \end{bmatrix} \Rightarrow -3, -\frac{6}{5}, 0$$

$$(J = 2, T = 0) \quad [1p_{1/2}1p_{3/2}] : \begin{bmatrix} -\frac{6}{5} \end{bmatrix} \Rightarrow -\frac{6}{5}$$

$$(J = 3, T = 0) \quad [(1p_{3/2})^2] : \begin{bmatrix} -\frac{6}{5} \end{bmatrix} \Rightarrow -\frac{6}{5}$$

- States at zero energy are spatially anti-symmetric. The matrix elements of the delta interaction vanishes for such states. SU(4) is at the origin of the observed degeneracies.
- For any choice of sign of Δ, the 0_1^+ level with $T = 1$ is below the 1_1^+ level with $T = 0$. For $\Delta < 0$ the order of the multiplet is $1^+, 2^+, 3^+$; for $\Delta > 0$ (not too large) the order is reversed, $3^+, 2^+, 1^+$.
- Observed $T = 0$ energies in ^{6}Li: 0 (1^+), 2.186 (3^+), 4.312 (2^+) and 5.650 (1^+). Observed $T = 1$ energies in ^{6}Li: 3.563 (0^+) and 5.366 (2^+). The observed splitting of the 0_1^+ ($T = 1$) and the 2_2^+ ($T = 1$) levels fixes $G_{T=1} \approx 1$ MeV. The observed splitting of the 0_1^+ ($T = 1$) and the 1_1^+ ($T = 0$) levels fixes $G_{T=0} \approx 2$ MeV. The observed splitting of the $3^+, 2^+, 1^+$ multiplet with $T = 0$ fixes $\Delta \approx 2$ MeV. We then obtain the following calculated energies in ^{6}Li. $T = 0$: 0 (1^+), 2.466 (3^+), 4.466 (2^+) and 4.866 (1^+); $T = 1$: 2.494 (0^+) and 4.286 (2^+).